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KURTOSIS, SKEWNESS AND NON-GAUSSIAN COSMOLOGICAL DENSITY PERTURBATIONS *

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Abstract

Cosmological topological defects as well as some non-standard inflation models can give rise to non-Gaussian density perturbations. Skewness and kurtosis are the third and fourth moments that measure the deviation of a distribution from a Gaussian. Measurement of these moments for the cosmological density field and for the microwave background temperature anistropy can provide a test of the Gaussian nature of the primordial fluctuation spectrum. In the case of the density field, the importance of measuring the kurtosis is stressed since it will be perserved through the gravitational evolution. Current constraints on skewness and kurtosis of primeval perturbations are obtained from the observed density constrast $\frac{\delta\rho}{\rho}$ on small scales and from recent COBE observations of temperature anistropies on large scales. It is also shown how, in principle, future microwave anistropy experiments might be able to reveal the initial skewness and kurtosis. It is shown that present data argue that if the initial spectrum is adiabatic, then it is probably Gaussian, but non-Gaussian isocurvature fluctuations are still allowed and these are what topological defects provide.

1. Introduction

The cosmic structure formation problem is in some sense an initial value problem: how did the universe generate the initial perturbations? In particular, one can divide initial condition models into two clear classes: Gaussian or non-Gaussian. Currently, the most popular model is the cold dark matter (CDM) model (Davis et al. 1985; Kolb & Turner 1989) where the zero-point quantum flunctuations in the inflaton field generate Gaussian primeval density perturbation with roughly equal power on all scales (after matter domination, small density fluctuations grow linearly with the cosmological expansion until $\delta\rho/\rho \sim 1$.) However, recent observations, including the extra large scale power seen in the APM 2-point correlation (Efstathiou 1991) and possibly the clustering of clusters and superclusters up to 100 Mpc scale (Bahcall & Soniera 1983; Tully 1986), coherent structure at scales $\sim$ 100 Mpc across, such as the Great wall (Geller & Huchra 1989), large voids (Kirshner et al. 1987; Geller & Huchra 1989) and the large scale velocity flows (Dressler et al. 1986; Collins et al. 1986), seem to be in conflict with the spectral assumption of the CDM model. However, it is also noted that on angular scales beyond those where galaxy structure are well observed, the "CDM model" is consistent with the COBE anistropy results (Smoot et al. 1992). One key ingredient of the standard CDM scenario is that the initial density field is Gaussian. However, on angular scales of 1° to 2°, the microwave anistropy limits are already marginally in confilict with Gaussian models for generating the velocity flows (Gorski et al. 1992).

A non-Gaussian initial density perturbation will not only help to fit the galaxy observations, but also point to an alternate scenario for generating the initial perturbation and yet might still satisfy the COBE anistropy measurements. In fact, the density field generated by some non-standard inflation scenarios (Bardeen et al. 1987; Salopek et al. 1989; Silk & Turner 1986) and topological defects (Vilenkin 1985; Turok 1989; Hill et al. 1989) can be characterized as non-Gaussian. Thus, the confirmation of a non-Gaussain density field would require new physics that yield topological defects or special non-standard inflation

in the early universe. Such a test of the Gaussian nature of the initial condition is, thus, very important and timely in today's cosmology.

In this paper, we discuss two ways to do this test. One is from the statistics of the galaxy counts in a redshift survey. However, since the density field we observe today has already gone through the "black-box" of non-linear gravitational evolution, one has to filter out this effect carefully to get a reliable estimate. Another is from the cosmic microwave background radiation (CMBR) anistropy experiments which measure the primeval density perturbations in baryons at redshift $z \sim 1000$. The density constrast is fairly small at this epoch. The Gaussian nature of the microwave background fluctuation reflects directly the nature of the primordial density. This approach is promising, especially after COBE reported the large scale temperature anistropy (Smoot et al. 1992) and data of a whole-sky map of the temperature anistropy are accumulating as are data on smaller scale anistropies (Experimental Papers 1992).

This paper is organized as follows: In part 2, we discuss skewness and kurtosis for adiabatic and isocurvature initial density perturbations as well as note how fractal initial fluctuation patterns behave. We present prototype models as examples. In part 3, we present a real space analysis of gravitational evolution, which is relevent to the measurement of skewness and kurtosis through statistics of galaxy counts in a redshift survey. In part 4, a k—space analysis is presented, which is the basis for constraining and measuring skewness and kurtosis through power spectrum and cosmic microwave background experiments. In part 5, we discuss how to measure initial skewness and kurtosis for adiabatic perturbations through the third and fourth order temperature correlation functions in cosmic microwave background experiments.

2. Skewness and Kurtosis

What are skewness and kurtosis? Wick's theorem states that for a random Gaussian

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variable ϕ with zero mean,

$$< \phi >= 0, < \phi...\phi_{(odd \ number)} >= 0;$$

$$< \phi_1 \phi_2 >= \xi(r_1 - r_2); \tag{2.1}$$

and $< \phi_1 \phi_2 \phi_3 \phi_4 >=< \phi_1 \phi_2 >< \phi_3 \phi_4 > + < \phi_1 \phi_3 >< \phi_2 \phi_4 > + < \phi_1 \phi_4 >< \phi_2 \phi_3 > .$

The departure from a Gaussain behavior can be characterized by the appearance of the skewness S and kurtosis K which are defined as the third and fourth moment of ϕ:

$$S =< \phi^3 >, \quad K =< \phi^4 > -3 < \phi^2 >^2 . \tag{2.2}$$

In $k-$space, the skewness S and kurtosis K are defined through

$$S =< \delta_{k_1} \delta_{k_2} \delta_{-k_1-k_2} >, k_1, k_2 \neq 0; k_1 + k_2 \neq 0. \tag{2.3}$$

$$K =< \delta_{k_1} \delta_{k_2} \delta_{k_3} \delta_{-k_1-k_2-k_3} >, k_1, k_2, k_3 \neq 0; k_1 + k_2 + k_3 \neq 0. \tag{2.4}$$

We study non-Gaussian density perturbations in the following three models:

(2.1) Quadratic model.

The density contrast δ is the superposition of a linear and quadratic function of a Gaussian variable:

$$\delta = \phi + \alpha(\phi^2 - < \phi^2 >), \tag{2.5}$$

where ϕ is a Gaussian. Motivated by Bardeen's two-field inflation model (Fan & Bardeen 1992), this model is ideal for studying primordial adiabatic non-Gaussian density perturbation. The distribution function $P(\delta)$ is found through $P(\delta)d\delta = P(\phi)d\phi$, or

$$P(\delta) = \frac{1}{\sqrt{4\alpha\delta + 1 + 4\alpha^2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma} \cdot e^{-\frac{1}{8\sigma^2\alpha^2}(\sqrt{4\alpha\delta+1+4\alpha^2\sigma^2}-1)^2}, \tag{2.6}$$

which is plotted in Fig. (1). In real space,

$$< \delta^2 >=< \phi^2 > +2\alpha^2 < \phi^2 >^2,$$

$$S = 6\alpha < \phi^2 >^2 +8\alpha^3 < \phi^2 >^3, \tag{2.7}$$

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$$\text{and} \quad K = 28\alpha^2 <\phi^2>^3 +48\alpha^4 <\phi^2>^4 .$$

The limiting case where $\alpha \ll 1$ is an ideal model for studying an initially skewed spectrum. To the lowest order,

$$<\delta^2> \rightarrow <\phi^2>,$$

$$S \rightarrow 6\alpha <\phi^2>^2, \tag{2.8}$$

$$\text{and} \quad K \rightarrow 0.$$

In $k-$space, by defining $P_0(k) = <\delta_k \delta_{-k}>$, the power spectrum, bispectrum (skewness) and the kurtosis are:

$$P(k) = P_0(k) + 2\alpha^2 \int P_0(k')P_0(k-k')d^3k' \rightarrow P_0(k)$$

$$S(k_1, k_2) = \alpha(P_0(k_1)P_0(k_2) + P_0(k_1)P_0(k_1+k_2) + P_0(k_2)P_0(k_1+k_2))$$

$$+8\alpha^3 \int P_0(k')P_0(k_1-k')P_0(k_2-k')d^3k' \tag{2.9}$$

$$\rightarrow \alpha(P_0(k_1)P_0(k_2) + P_0(k_1)P_0(k_1+k_2) + P_0(k_2)P_0(k_1+k_2))$$

$$\text{and} \quad K(k_1, k_2, k_3) \rightarrow 0.$$

(2.2) $O(N)$ non-linear σ-model.

When a $O(N)$ global symmetry is broken to O(N-1) by a N-component real scalar field ϕ in the early universe, the dynamics of the scalar fields, which is well described by $O(N)$ σ-model, has very interesting cosmological consequences. This model has been investigated in detail by Turok & Spergel (1991), and by Davis et al. (1992). It is a prototype model for producing non-Gaussian isocurvature density perturbations. In the standard cold dark matter dominated universe, the induced peturbations in matter are given by

$$\delta(r) = \frac{4\pi G\eta_0^2}{10} \int^{\eta_0} d\eta \partial_i \Theta_{0i}, \tag{2.10}$$

where $\Theta_{0i} = \frac{1}{a^2}\vec{\phi}\partial_i\vec{\phi}$. In the large N limit, ϕ and $\partial_i\phi$ are two independent Gaussian variables. Thus δ is Gaussian in this limit. The deviation from a Gaussian can be calculated from a $\frac{1}{N}$ expansion, which

in real space is:

$$S \sim (\frac{1}{\sqrt{N}}) < \delta^2 >^{3/2}$$

$$\text{and} \quad K \sim (\frac{12}{N}) < \delta^2 >^2; \tag{2.11}$$

in $k-$ space:

$$P(k) \sim \frac{1}{N} \int \frac{d^3 k^{'}}{(2\pi)^3} I_0(k^{'}) I_0(k - k^{'}),$$

$$S(k_1, k2) \sim \frac{1}{N^2} \int \frac{d^3 k^{'}}{(2\pi)^3} I_0(k^{'}) I_0(k_1 - k^{'}) I_0(k_2 - k^{'})$$

$$\text{and} \quad K(k_1, k2, k3) \sim \frac{1}{N^3} I_0(k^{'}) I_0(k^{'} - k_1) I_0(k^{'} + k_2) I_0(k^{'} - k_1 - k_3), \tag{2.12}$$

where $I_0(k) = < \dot{\phi}_k \dot{\phi}_{-k} >$.

(2.3) A sample fractal model.

This is an artificial model where

$$P(k_1, k_2, ..., k_n, -k_1 - k_2 - .. - k_n) = C_n P(k_1) P(k_2)...P(k_n) \tag{2.13}$$

where C_n is a constant. The reason we term it a "fractal model" is that, for a power law P(k), not only the two-point correlation function, but the n-point correlation function is invariant under scale transformation. Of course we know that physical mechanisms for generating fractal conditions are bounded by causality. Thus, fractals on scales larger than the horizon at the time of structure formation (Luo & Schramm, 1992) are not expected.

3. Real Space Analysis

It is convenient to normalize the skewness and kurtosis by the variance. In this section, we define the normalized skewness α and kurtosis β as:

$$\alpha = \frac{< \phi^3 >}{< \phi^2 >^{3/2}}, \quad \beta = \frac{< \phi^4 >}{< \phi^2 >^2} - 3. \tag{3.1}$$

In structure formation theory, the density contrast δ is used as a random variable. For a Gaussian initial perturbation, both α and β vanish.

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The evolution of δ in an expanding Einstein-de Sitter universe is (Peebles 1980):

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G\rho_b\delta(1+\delta) + \nabla\delta\cdot\nabla\phi/a^2 + \partial_\alpha\partial_\beta(v^\alpha v^\beta)/a^2, \qquad (3.2)$$

where

$$\phi = -G\rho_b a^2\Delta(x); \Delta(x) = \int\frac{dx'\delta(x')}{|\bar{x}-\bar{x}'|}; v^\alpha = \frac{a}{4\pi D}\dot{D}\Delta_{,\alpha}$$

In the linear regime,

$$\delta = \delta_0 \approx \delta_i D(t); \qquad (3.2)$$

where $D(t) = (t/t_i)^{2/3}$ is the growing mode in the matter dominated universe, t_i is approximately the recombination time when the structure starts to grow, and δ_i is the initial density perturbation. The above equation shows that in the linear regime, spectrum features of the initial perturbation are all preserved. The non-linear effect can be found by expanding the density contrast to higher order:

$$\delta = \delta_0(1 + \epsilon(x,t)); \delta_0 \ll 1; \epsilon \ll 1. \qquad (3.4)$$

Then

$$\ddot{\epsilon} + 2(\frac{\dot{D}}{D} + \frac{\dot{a}}{a})\dot{\epsilon} = 4\pi G\rho_b\delta_0 - G\rho_b\frac{\delta_{0,\alpha}}{\delta_0}\Delta_{,\alpha} + \frac{\dot{D}^2}{16\pi^2 D^2\delta_0}\partial_\alpha\partial_\beta(\Delta_{,\alpha}\Delta_{,\beta}), \qquad (3.5)$$

which gives

$$\delta = \delta_0 + \frac{5}{7}\delta_0^2 - \frac{1}{4\pi}\delta_{0,\alpha}\Delta_{,\alpha} + \frac{1}{56\pi^2}\Delta_{,\alpha\beta}\Delta_{,\alpha\beta}. \qquad (3.6)$$

From this we can calculate the skewness developed from non-linear gravitational evolution effects (Peebles, 1980):

$$< \delta^3 > = \frac{34}{7} < \delta_0^2 >^2, \quad \text{or,} \quad \alpha = \frac{34}{7}\sqrt{< \delta_0^2 >}. \qquad (3.7)$$

Notice that the normalized skewness induced by the gravitational evolution scales as the square root of the variance. However, as shown by Scherrer and Bertschinger (1991), the normalized skewness of the discrete-mass seeded model is constant. Several authors (Silk & Juszkiewicz 1991; Coles & Frenk 1991) have used this argument to attempt to discriminate an initial Gaussian from a non-Gaussian perturbation. But one point can be

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lost in such an argument: the non-Gaussian initial perturbation also must undergo non-linear gravitational evolution such that all of the initial skewness can be washed away. We can redo Peebles's evolution calculation under the condition that the initial perturbation is non-Gaussian (Lahav et al 1992). For the topological defect seeded model, we consider the case where α and β are constant initially.

$$< \delta^2 > = < \delta_0^2 > + 2 < \frac{5}{7}\delta_0^3 - \frac{\delta_0}{4\pi}\delta_{0,\alpha}\Delta_{,\alpha} + \frac{\delta_0}{56\pi^2}\Delta_{,\alpha\beta}\Delta_{,\alpha\beta} >$$

$$= < \delta_0^2 > + < \delta_0^3 > = < \delta_0^2 > (1 + \alpha\sqrt{< \delta_0^2 >}); \qquad (3.8)$$

$$< \delta^3 > = < \delta_0^3 > + 3 < \frac{5}{7}\delta_0^4 - \frac{1}{4\pi}\delta_0^2\delta_{0,\alpha}\Delta_{,\alpha} + \frac{\delta_0^2}{56\pi^2}\Delta_{,\alpha\beta}\Delta_{,\alpha\beta} >$$

$$= < \delta_0^3 > + \frac{34}{21} < \delta_0^4 > = \alpha < \delta_0^2 >^{3/2} + (\frac{34}{7} + \frac{\beta}{3}) < \delta_0^2 >^2 . \qquad (3.9)$$

Thus, the overall skewness will be:

$$\alpha_{\text{non-Gaussian}} = \frac{< \delta^3 >}{< \delta^2 >^{3/2}} = \frac{\alpha + (\frac{34}{7} + \frac{\beta}{3})\sqrt{< \delta_0^2 >}}{(1 + \alpha\sqrt{< \delta_0^2 >})^{3/2}} \qquad (3.10)$$

The observed skewness vs. the initial skewness is plotted in Fig. 2, assuming the density contrast $\frac{\delta\rho}{\rho} = \sqrt{< \delta_0^2 >} = \sqrt{\sigma} = 0.5$, where σ is the standard variance of δ_0. The maximum deviation from a Gaussian is $< 20\%$, which is hard to discriminate using present galaxy count statistics analysis (although future surveys such as the 10^6 redshift digital sky survey may improve this possibility).

However, the kurtosis of an evolved Gaussian distribution remains zero even to second order perturbation theory. This is shown below:

$$< \delta^4 > = < \delta_0^4 > + 4 < \delta_0^4 \epsilon >$$

$$= < \delta_0^4 > + 4 < \frac{5}{7}\delta_0^5 - \frac{1}{4\pi}\delta_0^3\delta_{0,\alpha}\Delta_{,\alpha} + \frac{\delta_0^3}{56\pi^2}\Delta_{,\alpha\beta}\Delta_{,\alpha\beta} > . \qquad (3.11)$$

According to Eq. (1), all the odd moments of a Gaussian variable vanish, so

$$< \frac{5}{7}\delta_0^5 - \frac{1}{4\pi}\delta_0^3\delta_{0,\alpha}\Delta_{,\alpha} + \frac{\delta_0^3}{56\pi^2}\Delta_{,\alpha\beta}\Delta_{,\alpha\beta} > = 0, \qquad (3.12)$$

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which gives:

$$\beta = \frac{<\delta^4>}{<\delta^2>^2} - 3 = \frac{<\delta_0^4>}{\sigma^2} - 3 = 0. \tag{3.13}$$

Although full non-linear numerical analysis (Dekel & Yahil 1992) can generate small non-zero kurtosis even for Gaussians, the fact that it comes in only at high order (see Eq. (3.13)) keeps it small relative to the non-Gaussian case described below. If the initial distribution is non-Gaussian, from Eq. (3.8) and (3.11),

$$<\delta^2> = \sigma(1 + \alpha\sqrt{\sigma}); \tag{3.14}$$

$$<\delta^4> = <\delta_0^4> +3 <\delta_0^5> \approx <\delta_0^4>. \tag{3.15}$$

Moments higher than 4 are generally small in all topological defect models, so we neglect them in this paper. The observed kurtosis for a non-Gaussian field is:

$$\beta_{\text{non-Gaussian}} = \frac{<\delta^4>}{<\delta^2>^2} - 3 = \frac{3 + \beta}{(1 + \alpha\sqrt{\sigma})^2} - 3. \tag{3.16}$$

We plot the observed kurtosis vs initial kurtosis in Fig. 3. Again, we assume that the density contrast in linear theory $\frac{\delta\rho}{\rho} = \sqrt{\sigma} = 0.5$. The difference between a Gaussian and non-Gaussian is obvious. Furthermore, the observed kurtosis is a sensitive function of the initial skewness as shown in the Fig. 3. Contrary to naive expectation, a large observed kurtosis ($\gg 0.5$) doesn't mean that the initial kurtosis is large; rather. it suggests that the initial distribution has a negative skewness.

Much interesting information on the initial distribution is contained in the sign of kurtosis. For example. it would be very interesting if a negative kurtosis is discovered. There are some sharp inequalities between the skewness and kurtosis. The Cauchy-Schwarz inequality implies that

$$\beta \geq \alpha^2 - 3 \tag{3.17}$$

for any given distribution. This gives an absolute lower bound for the kurtosis. If we expect the initial density distribution to be infinitely divisible,* then we have (Rohatgi &

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* An infinitly divisible distribution is a generalization of a Gaussian. Most of the

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Szekely 1989):

$$\beta \geq \alpha^2. \tag{3.18}$$

In this case, the minimun of the kurtosis will be zero when the distribution is symmetric so that $\alpha = 0$. Thus, a negative kurtosis will rule out not only a Gaussian or Poisson distribution, but all the possible translations of the convolution of the Gaussian and Poisson distribution. In this case, inflation induced random quantum flunctuations would be ruled out as the initial perturbations for structure formation. We are forced to find something more exotic like topological defects. (Of course, the amplitude of the inflation-induced fluctuations may be too low to cause a observable effect.)

4. k—Space Analysis

Newtonian equations are valid for describing the evolution of density peturbation when the k-mode is inside the horizon (Peebles 1980):

$$\partial_t \delta + \frac{1}{a} \nabla \cdot [\vec{v}(1 + \delta)] = 0,$$

$$\partial \vec{v} + \frac{1}{a} \vec{v}(\nabla \vec{v}) + \frac{\dot{a}}{a} \vec{v} + \frac{1}{a} \nabla \phi = 0, \tag{4.1}$$

$$\nabla^2 \phi = 4\pi G a^2 \rho_0 \delta,$$

where a is the expansion factor of the background FRW universe. The equations can be solved approximately by expanding δ in a power series (Juszkiewicz 1981; Vishniac 1983):

$$\delta(k,t) = \delta_1(k,t) + \delta_2(k,t) + \dots \tag{4.2}$$

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distributions used in cosmology today are infinitly divisible, such as a Gaussian, Poisson, exponential, negative-binomial, compound Poisson etc. The physical significance of the infinitly divisible density distribution is that each volume is stachostically independent (Saslaw 1989). Mathematically, it can be represented as the distribution of the sum $S_n = X_{1,n} + \dots + X_{n,n}$ of n independent random variables with a common distribution. In the Levy-Khintchine representation (Feller 1966), it can be approximated by translation of convolution of Gaussian and Poisson distribution.

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where

$$\delta_1(k,t) = \alpha_k(t/t_i)^{2/3},$$

$$\delta_2(k,t) = \frac{1}{14} \int d^3k' J(k,k',k-k')\alpha_{k'}\alpha_{k-k'}, \tag{4.3}$$

where

$$J(k,l,m) = 2k^2(\vec{l}\vec{m})(lm)^{-2} + 5(\vec{k}\cdot\vec{l})l^{-2}. \tag{4.4}$$

The power spectrm of the density peturbation is given by:

$$P(k,t) = |\delta(k,t)|^2$$

$$= |\delta_1(k,t)|^2 + 2Re[\delta_1^*(k,t)\delta_2(k,t)] + |\delta_2(k,t)|^2 + 2Re[\delta_1^*(k,t)\delta_3(k,t)] + ... \tag{4.5}$$

We assume that the universe has critical density $\Omega = 1$ so that $a(t) \sim t^{2/3}$. Assuming the initial density power spectrum is $P_0(k) = < \alpha_k\alpha_{-k} >$, then

$$P(k,t) = P_0(k)(t/t_i)^{4/3} + P_S(k)(t/t_i)^2 + [P_{22}(k) + P_{13}(k) + P_K(k)](t/t_i)^{8/3} + ... \tag{4.6}$$

where

$$P_S(k) = \frac{k^2}{7 \times (2\pi)^2} \int k' S(k,k')dk' \int_{-1}^{+1} dy \frac{(3-10y^2)k' + 7ky}{k^2 - 2kk'y + k'^2} \tag{4.7}$$

$$P_K(k) = \frac{1}{14^2(2\pi)^6} \int K(k,k_1,k_2)d^3k_1 d^3k_2 \int_{-1}^{+1} dy \frac{(3-10y^2)k_1 + 7ky}{k^2 - 2kk_1y + k_1^2} \frac{(3-10y^2)k_2 + 7ky}{k^2 - 2kk_2y + k_2^2}. \tag{4.8}$$

$P_S(k)$ and $P_K(k)$ are the contributions from the non-Gaussian nature of the initial density perturbation. They vanish for a Gaussian field. $P_{22}(k), P_{13}(k)$ are given by Suto and Sasaki (1991). For the standard CDM spectrum $P_0(k)$,

$$P_0(k) = A(k/k_{eq}), \text{ if } k < k_{eq}; \quad P_0(k) = A(k/k_{eq})^3, \text{ if } k > k_{eq}. \tag{4.9}$$

All of the power spectrum can be calculated analytically. However, for the skewness and kurtosis given by quadratic models and the $O(N)$ σ model, the expressions appear too complicated to be of any immediate practical usefulness. Thus, we show here only the results from the fractal model:

$$P_S(k) = S\frac{A}{7(2\pi)^2}P_0(k)k_{eq}^3 f(q), \tag{4.10}$$

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$$P_K(k) \sim KP_0(k)[\frac{A}{7(2\pi)^2}P_0(k)k_{eq}^3 f(q)]^2 \qquad (4.11)$$

where $q = \frac{k}{k_{eq}}$. S, K are the constants C_2, C_3 in Eq(2.13) and

$$f(q) = q^2(-\frac{5}{3} - \frac{4}{q^2} + \frac{4q^4 + 6q^2 + 13q + 6}{3q^3}\ln|\frac{q+1}{q-1}| + q^2 \ln|\frac{q^2-1}{q^2}|). \qquad (4.12)$$

With these tools in hand, we can move on to study the bounds on skewness and kurtosis through CMBR and the power spectrum observed from redshift surveys.

There are several ways in which the CMBR might be perturbed. The primary effect is the gravitation potential ϕ at the last scattering surface (Sachs & Wolfe 1967), where

$$\frac{\delta T}{T} = \phi/3. \qquad (4.13)$$

After the photon leaves the last-scattering surface, it can also be perturbed by the secondary effects, such as going through a time-varying gravitational potential (as would occur in a late-time phase transition (Hill et al. 1989)), where

$$\frac{\delta T}{T} = 2\int_{photon\ path} \partial_t \phi dt. \qquad (4.14)$$

The gravitational potential is related to the density perturbation through the Poisson equation:

$$\nabla^2 \phi = 4\pi G a^2 \rho \delta \qquad (4.15)$$

or:

$$\phi(k,t) = \phi_1(k,t) + \phi_2(k,t) + ...$$
$$= -\frac{4\pi G\rho}{2k^2}(\delta_1(k)a(t) + \delta_2(k)a^2(t)) \qquad (4.16)$$

where δ_1 and δ_2 are given in the previous section. To first order,

$$\phi(k,t) = \phi_1(k,t) = -\frac{4\pi G\rho}{2k^2}\delta_1(k)a^3(t). \qquad (4.17)$$

Since $\rho \approx a^{-3}$ in the matter domination regime, ϕ is a constant. Thus, $(\frac{\delta T}{T})_{secondary} = 0$. This is the result which is stated by Peebles (1980) as:

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"The growing mode itself doesn't perturb the microwave background."

To the second order,

$$\phi = constant - \frac{4\pi G\rho}{2k^2}\delta_2(k)a^4(t)$$

$$= -\frac{3H_0^2}{2k^2}\delta_2 a(t) \propto a(t).$$

Thus,

$$\left(\frac{\delta T}{T}\right)_k \sim \int \dot{a}dt\delta_2(k), \tag{4.18}$$

which depends only on the second order term in the perturbation series of δ. Since skewness is the cross term of the first and second term, thus we have:

"The skewness doesn't perturb the CMBR."

The kurtosis will perturb the CMBR through a second order effect. As shown by Gonzalez, Sanz, & Silk (1992), the effect on rms CMBR temperature anistropy is of order $K \times 10^{-6}$. COBE discovered the rms temperature at the level of 10^{-5} which thus puts a loose bound on kurtosis

$$K < 10, \tag{4.19}$$

which is satisfied by all initial distributions we consider.

The results above show that CMBR temperature fluctuations alone do not yet constrain the skewness and kurtosis of the initial perturbations. However, combining the COBE results and the density observed today at $\sim$ 10Mpc scale can constrain skewness and kurtosis.

Since we consider the case where the initial density perturbations are non-Gaussian, the evolution of the power spectrum will be different from that of a Gaussian perturbation. Initially, let us use a basic CDM power spectrum. In that case, from Eq. (4.5), the power spectrum observed today can be written as:

$$P(k) = P_0(k)\{1 + (1 + Z_i)P_S(k) + (1 + Z_i)^2 P_K(k)\} \tag{4.20}$$

where $P_0(k)$ is the power spectrum today predicted by CDM and $Z_i \approx 1000$ is the redshift at the recombination epoch. The density constrast at a given scale can be found through the convolution of the power spectrum and an approximate window function. The redshift surveys gives (Davis & Peebles 1983)

$$(\frac{\delta\rho}{\rho})(r = 8Mpc) \approx 1. \tag{4.21}$$

COBE reports (Smoot et al. 1992) that the quadrople temperature anistropy coefficient is

$$a_2 = (6 \pm 2) \times 10^{-6}, \tag{4.22}$$

which gives an overall normalization of the whole spectrum. Combining the two results, we have:

$$S\frac{96\pi(1 + Z_i)}{7(2\pi)^2}(a_2)^2(H_0^{-1}k_{eq})^4 f(q = k_{eq}r_0^{-1}) < 2 \tag{4.23}$$

$$K\frac{96\pi(1 + Z_i)}{7(2\pi)^2}(a_2)^4(H_0^{-1}k_{eq})^8 f^2(q = k_{eq}r_0^{-1}) < 2 \tag{4.24}$$

which gives

$$S < 0.01 \tag{4.25}$$

$$K < 10^{-4}. \tag{4.26}$$

This result strongly suggests that if the primordial perturbation is adiabatic, it is most probably Gaussian. If non-Gaussianality is found, it must be generated at a later time so that the power enhancement due to the gravitational evolution is small. A cautionary remark is that when density perturbation begins to go non-linear, our perturbation analysis may not be reliable. A detailed N-body simulation analysis will be necessary to check our conclusions here.

5. Testing Non-Gaussianality Through CMBR

When the primordial density perturbation is adiabatic, it is related to the temperature anistropy in CMBR through the Sachs-Wolfe effect:

$$\delta T/T = \phi/3, \tag{5.1}$$

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where ϕ is the gravitational potential at the last scattering surface,

$$\phi = G\rho_0 \int \frac{\delta(r')}{|r-r'|}d^3r' = \frac{3H_0^2}{8\pi}\delta(r)*W(r). \tag{5.2}$$

The last line of the equation is written in the convolution form where $W(r) = 1/r$. In $k-$space, we have

$$(\frac{\delta T}{T})_k = -\frac{H_0^2}{2k^2}\delta_k. \tag{5.3}$$

Thus, the two-point temperature correlation function is given by (Gouda et al. 1989):

$$C(\hat{r}) = \int < \frac{\delta T}{T}_k \frac{\delta T}{T}_{-k} > e^{i\vec{k}\vec{r}}d^3k = \frac{H_0^4}{8\pi}\int <\delta_k\delta_{-k}> j_0(kr)dk, \tag{5.4}$$

where $r = 2D\sin\theta/2$, θ is the angle between two beams, and $D = 2H_0^{-1}$ is the distance from us to the last scattering surface. The three and four point temperature correlation is given by:

$$S_T(\hat{r},\hat{s}) = -\frac{H_0^6}{8}\int \frac{<\delta_{k_1}\delta_{k_2}\delta_{-k_1-k_2}>^2}{k_1^2 k_2}|k_1+k_2|^2 e^{i(\vec{k_1}\cdot\vec{r1}+\vec{k_2}\cdot\vec{r2})}d^3k_1 d^3k_2; \tag{5.5}$$

$$K_T(\hat{r},\hat{s},\hat{t}) = \frac{H_0^8}{16}\int \frac{<\delta_{k_1}\delta_{k_2}\delta_{k_3}\delta_{-k_1-k_2-k_3}>}{k_1^2 k_2^2 k_3^2 |k_1+k_2+k_3|^2}e^{i(\vec{k_1}\cdot\vec{r1}+\vec{k_2}\cdot\vec{r2}+\vec{k_3}\cdot\vec{r3})}d^3k_1 d^3k_2 d^3k_3 x. \tag{5.6}$$

For Gaussian density perturbations,

$$S_T = 0, \tag{5.7}$$

$$K_T(\hat{r},\hat{s},\hat{t}) = \frac{H_0^8}{16}(C(|\hat{r}-\hat{s}|)\cdot C(\hat{t}) + C(\hat{r})\cdot C(|\hat{s}-\hat{t}|) + C(\hat{s})\cdot C(|\hat{r}-\hat{t}|)). \tag{5.8}$$

Considering the present error on the observational data, it is hard to obtain reliable kurtosis from the 4th order temperature correlation function. However, the 3-point correlation function is strictly zero for Gaussian models and the non-Gaussian models are not. Thus, to measure the 3-point temperature correlation function through the whole-sky COBE data becomes very important.

In conclusion, we have shown how to test the Gaussian nature of the primordial fluctuation through the measurement of the skewness and kurtosis for the cosmological density field and for the microwave background temperature anistropy. We conclude that present

data argue that if the initial spectrum is adiabatic, then, it is probably Gaussian, but non-Gaussian isocurvature fluctuations are still allowed. Topological defects are in general isocurvature in nature so they remain as viable alternatives at the present time.

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FIGURE CAPTIONS

Fig. 1: The probability distribution function $P(\delta)$ for the quadratic model. The solid line is for the Gaussian ($\alpha = 0$) case. The dotted line is for $\alpha = -0.1$ (negative skewed). The dashed line is for $\alpha = 0.1$ (positive skewed). The variance is the distributions ia chosen to be 1.

Fig. 2: The evolved skewness S.

Fig. 3: The evolved kurtosis K.